Fourier Transform

EECS 442

Fall 2023, University of Michigan

Linear Independence

- A set of vectors are linearly dependent if you can write one as a linear combination of the others

- Suppose: $\boldsymbol{a} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\boldsymbol{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$ $\boldsymbol{c} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 2\boldsymbol{a} \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$$

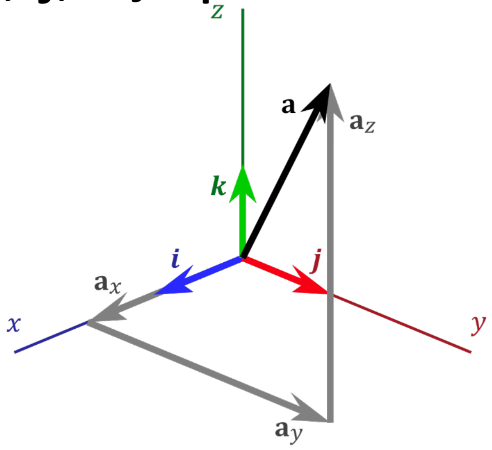
Is the set {a, b, c} linearly independent?

Is the set {a, b, x} linearly independent?

Basis

- Consider all vectors in $\mathbb{R}^3$ (3D Plane)
- A set of **linearly independent vectors** whose **span is the whole 3D plane** are called the basis for the 3D plane

E.g., the standard basis {i, j, k} spans the whole 3D plane:



Any other vector in the plane (e.g., a) is a linear combination of {i, j, k}

Using Basis for expressing vectors

Example: $$\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

{i, j, k} are the three basis vectors here

We could decompose it in terms of some other basis as well

Intuition behind Fourier transform: change of basis

$$\vec{f} = \overrightarrow{e_1} \cdot \mathrm{F}(0) + \overrightarrow{e_2} \cdot \mathrm{F}(1) + \overrightarrow{e_3} \cdot \mathrm{F}(2)$$

where

$$\vec{f} \triangleq f(n), \overrightarrow{e_1} \triangleq \frac{1}{3} e^{2\pi i n * \frac{0}{3}}, \overrightarrow{e_2} \triangleq \frac{1}{3} e^{2\pi i n * \frac{1}{3}}, \overrightarrow{e_3} \triangleq \frac{1}{3} e^{2\pi i n * \frac{2}{3}}$$

$$\vec{f} = \begin{bmatrix} | \\ f \\ | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ F(2) \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Bottom line:

- Fourier coefficients are coordinates in the Fourier basis defined by $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$
- Calculating Fourier coefficients is just about finding the projection on the vector $f(n)$ along the basis

Discrete Fourier Transform

We can extend this to any vector of length N:

$$F[u] = \sum_{n=0}^{N-1} f[n] e^{\left(-2\pi i \frac{un}{N}\right)}$$

where $e^{\left(-2\pi i \frac{un}{N}\right)} = \cos\left(2\pi i \frac{un}{N}\right) - i \sin\left(2\pi i \frac{un}{N}\right)$

- Output F is a weighted sum of sines and cosines with the weights governed by input f
- We can think of the **exponentials** as **basis functions**, and the function F is expressed in terms of those basis

Continuous Fourier Transform

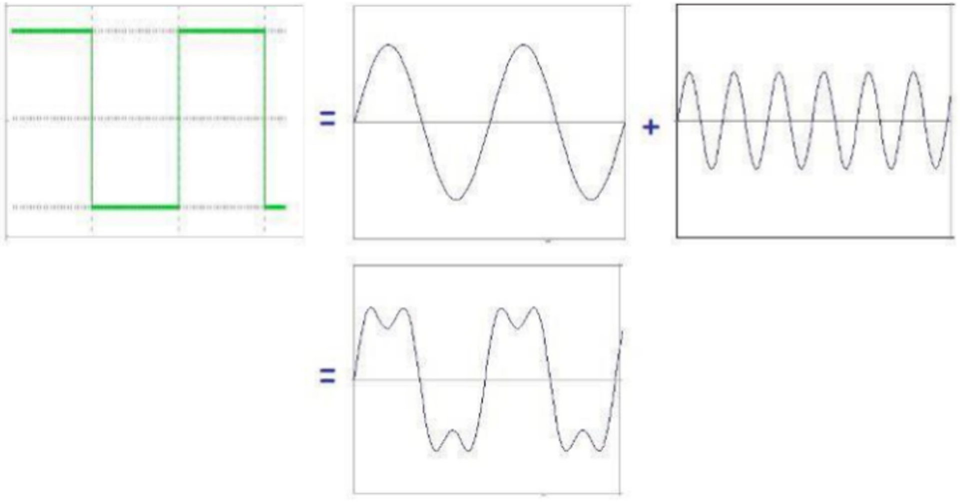
Going from continuous to discrete just means we take the integral from $-\infty$ to ∞:

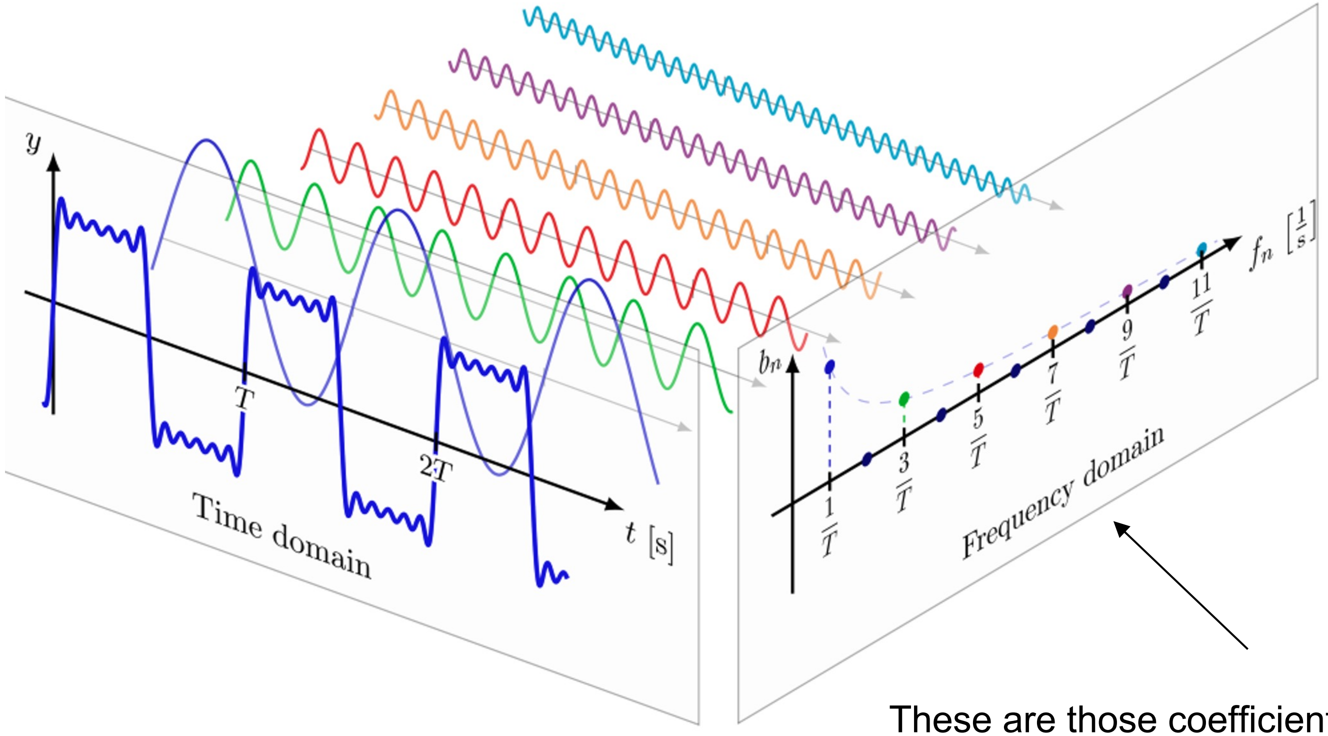
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt$$

Time domain to frequency domain

- Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies

    i.e., if we weighted sum across the different frequencies, we reconstruct the original signal



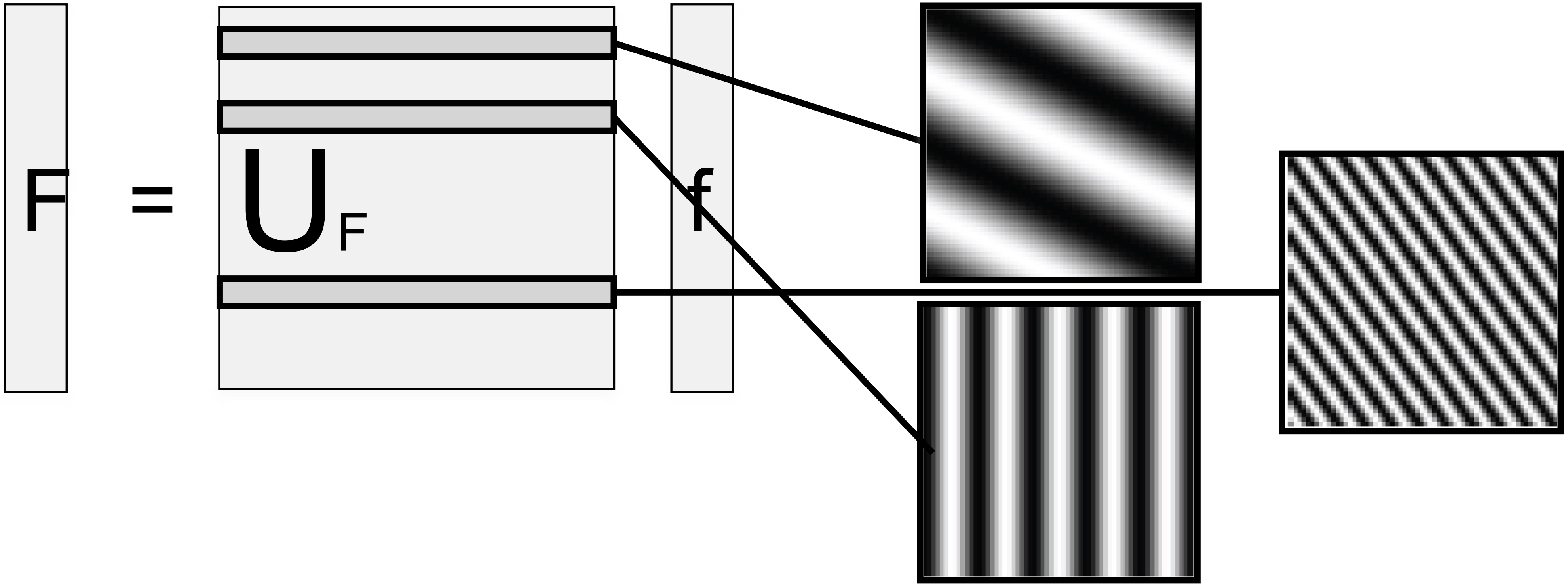


These are those coefficients!

Source: A.Efros

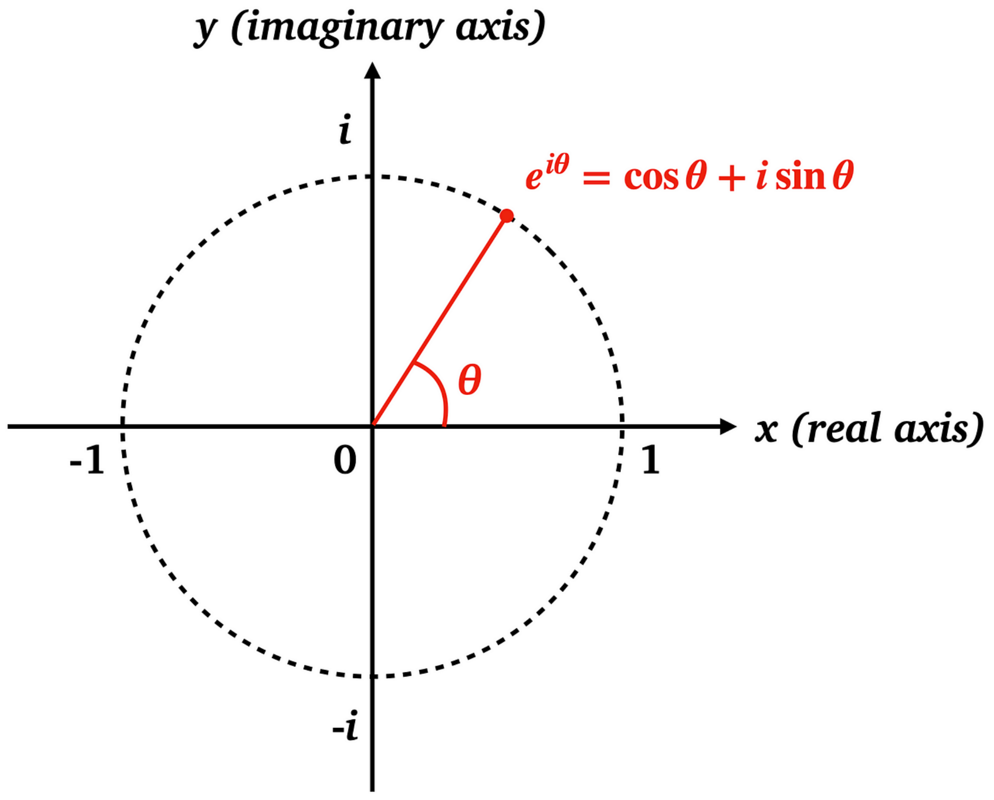
Frequency Basis

We’re using a basis of sinusoids with different frequencies.



Complex Exponential Review

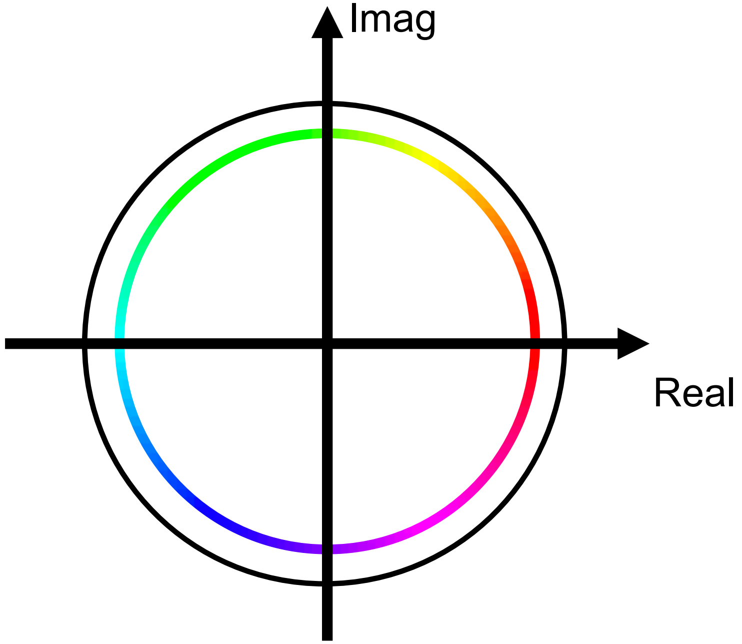
$$\vec{f} \triangleq f(n), \overrightarrow{e_1} \triangleq \frac{1}{3} e^{2\pi i n * \frac{0}{3}}, \overrightarrow{e_2} \triangleq \frac{1}{3} e^{2\pi i n * \frac{1}{3}}, \overrightarrow{e_3} \triangleq \frac{1}{3} e^{2\pi i n * \frac{2}{3}}$$

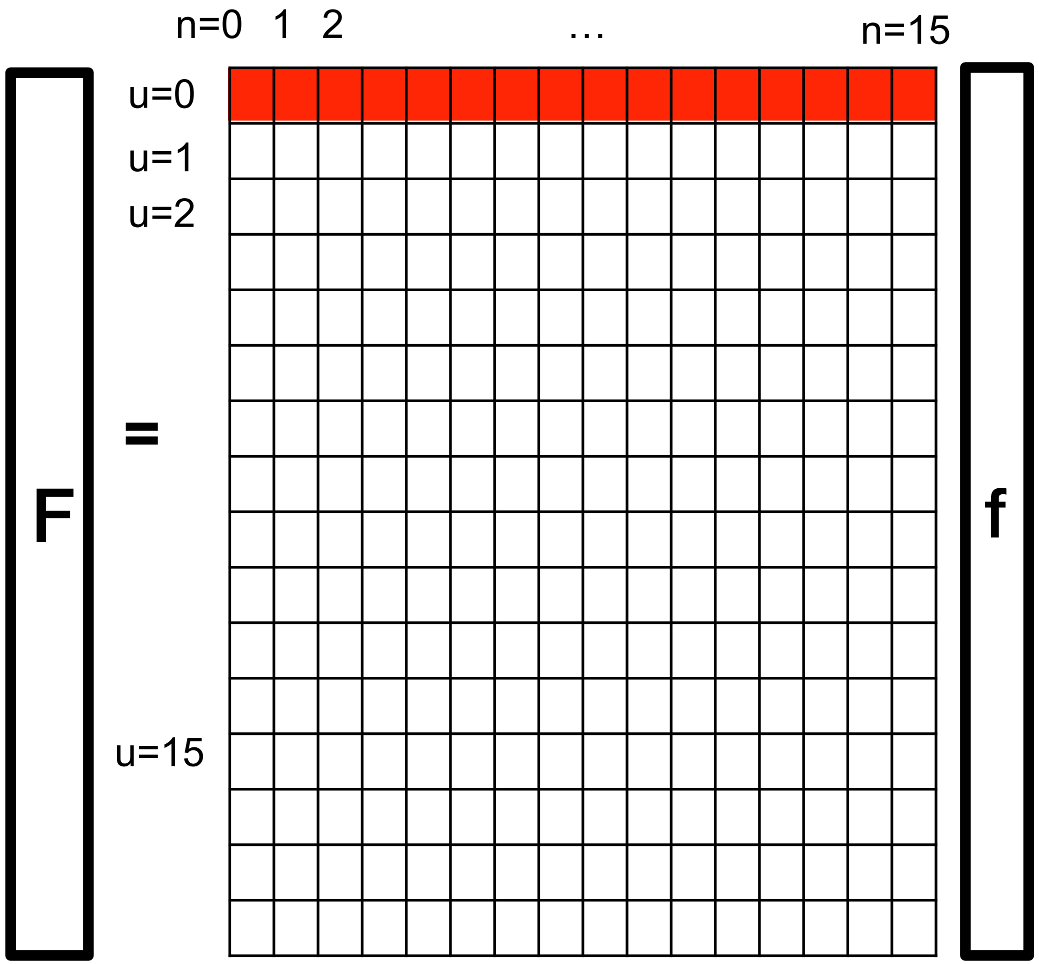


Unit circle in the complex plane

Visualizing Fourier Transform Matrix

$$\exp\left(-2\pi i \frac{un}{N}\right)$$ For N=16





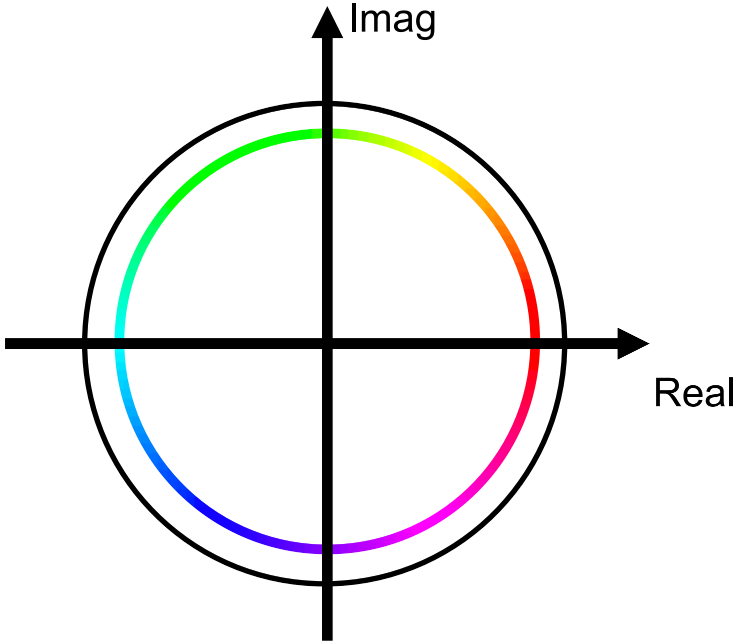
When u = 0, $\exp\left(-2\pi i \frac{un}{N}\right) = \exp(-2\pi i)$ for all n

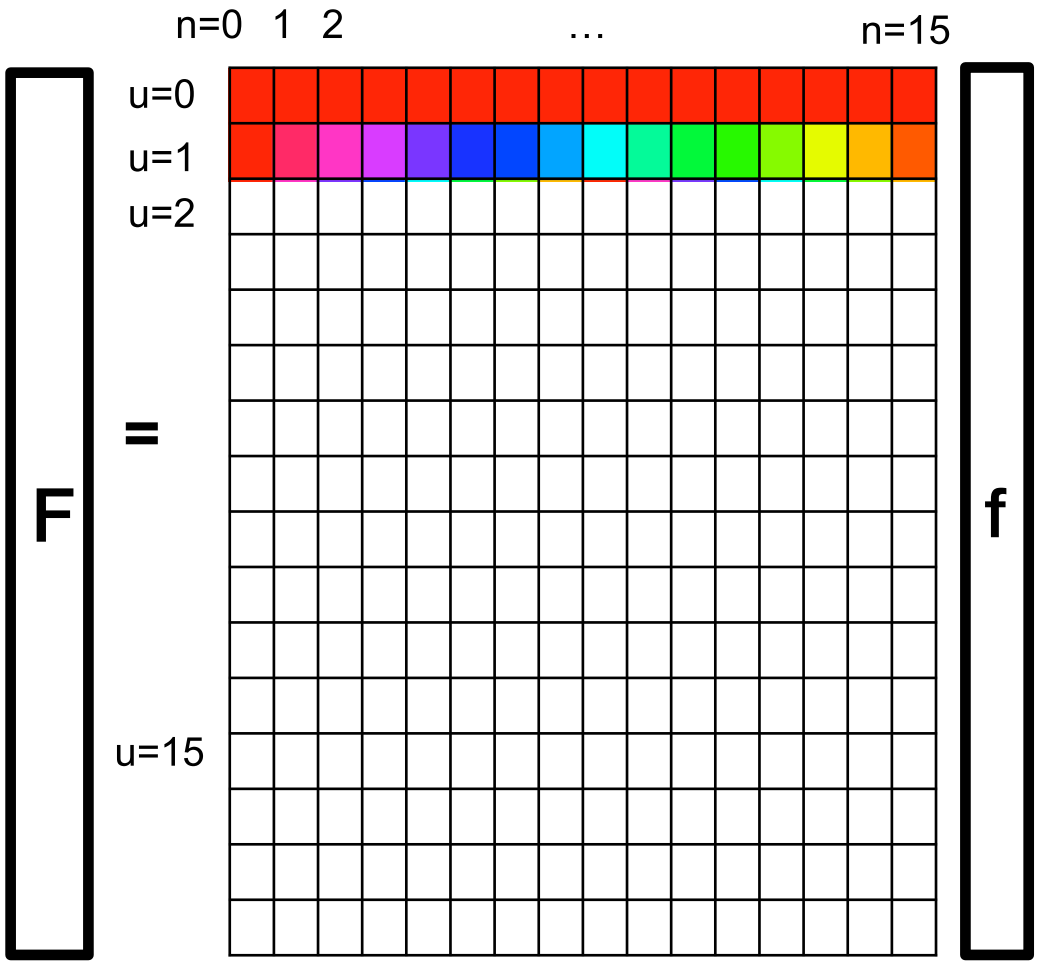
➔ no change in frequency from n = 0 to n = N

Source: Torralba, Freeman, Isola

Visualizing Fourier Transform Matrix

$$\exp\left(-2\pi i \frac{un}{N}\right)$$ For N=16





16x16 array

When u = 1, $\exp\left(-2\pi i \frac{un}{N}\right) = \exp\left(-2\pi i \frac{n}{N}\right)$ for all n

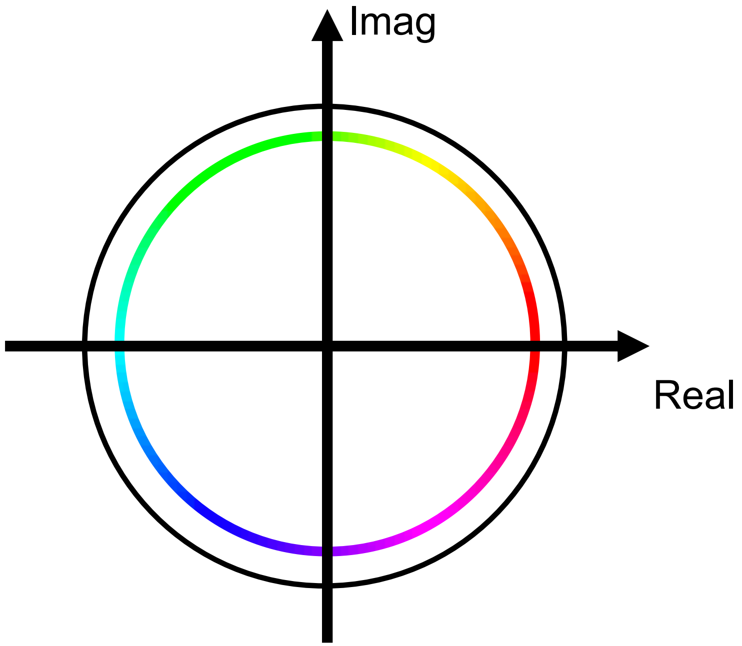
➔ no change in frequency from n = 0 to n = N

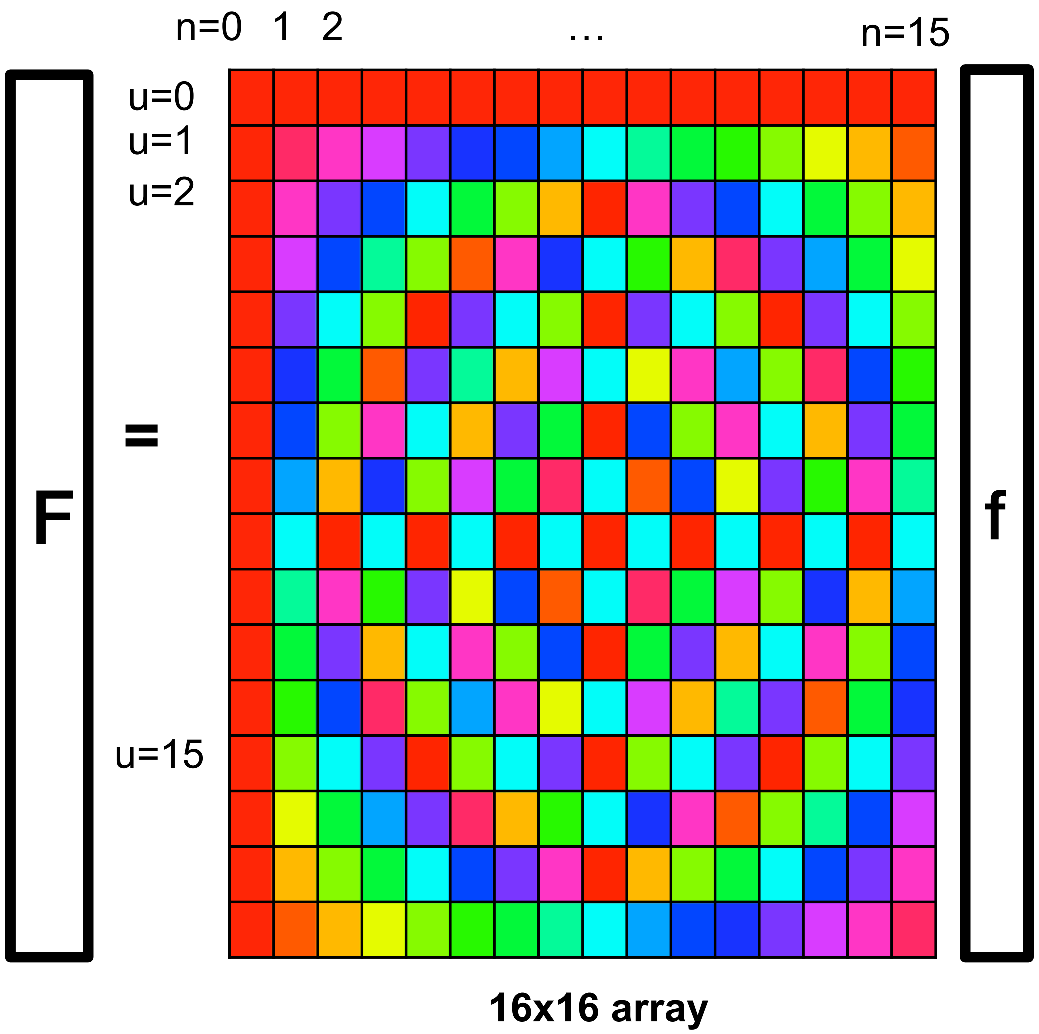
Source: Torralba, Freeman, Isola

Visualizing Fourier Transform Matrix

$$\exp\left(-2\pi i \frac{un}{N}\right)$$

For N=16





Source: Torralba, Freeman, Isola

Examples

Let’s say a = [1, 0, 0, 0], N = 4

$$F[u] = \sum_{n=0}^{N-1} f[n]e^{\left(-2\pi i\frac{un}{N}\right)} \qquad (u = 0, 1, \ldots N - 1)$$

Examples

Let's say a = [1, 0, 0, 0], N = 4

$$F[u] = \sum_{n=0}^{N-1} f[n]e^{\left(-2\pi i\frac{un}{N}\right)} \qquad (u = 0, 1, \ldots N-1)$$

$$F[u] = f[0]\, e^{\left(-2\pi i\frac{u*0}{4}\right)} + f[1]\, e^{\left(-2\pi i\frac{u*1}{4}\right)} + f[2]\, e^{\left(-2\pi i\frac{u*2}{4}\right)} + f[3]\, e^{\left(-2\pi i\frac{u*3}{4}\right)} = f[0] = 1$$

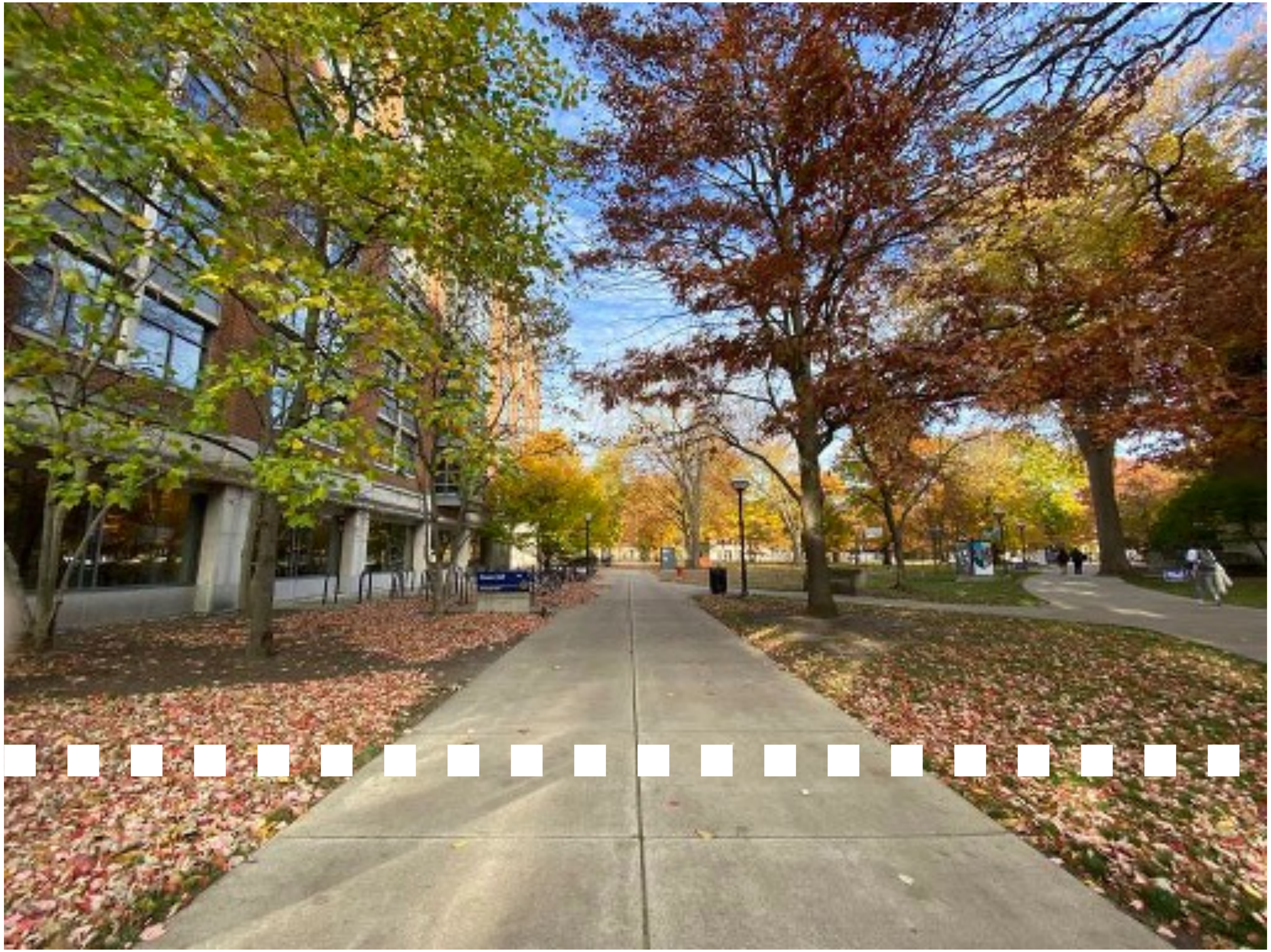
```
a = np.array([1,0,0,0])
np.fft.fft(a)
```

→

```
array([1.+0.j, 1.+0.j,
1.+0.j, 1.+0.j]
```

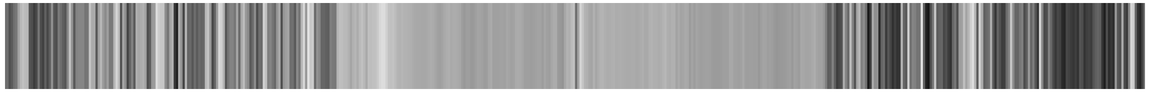
All coefficients are 1!

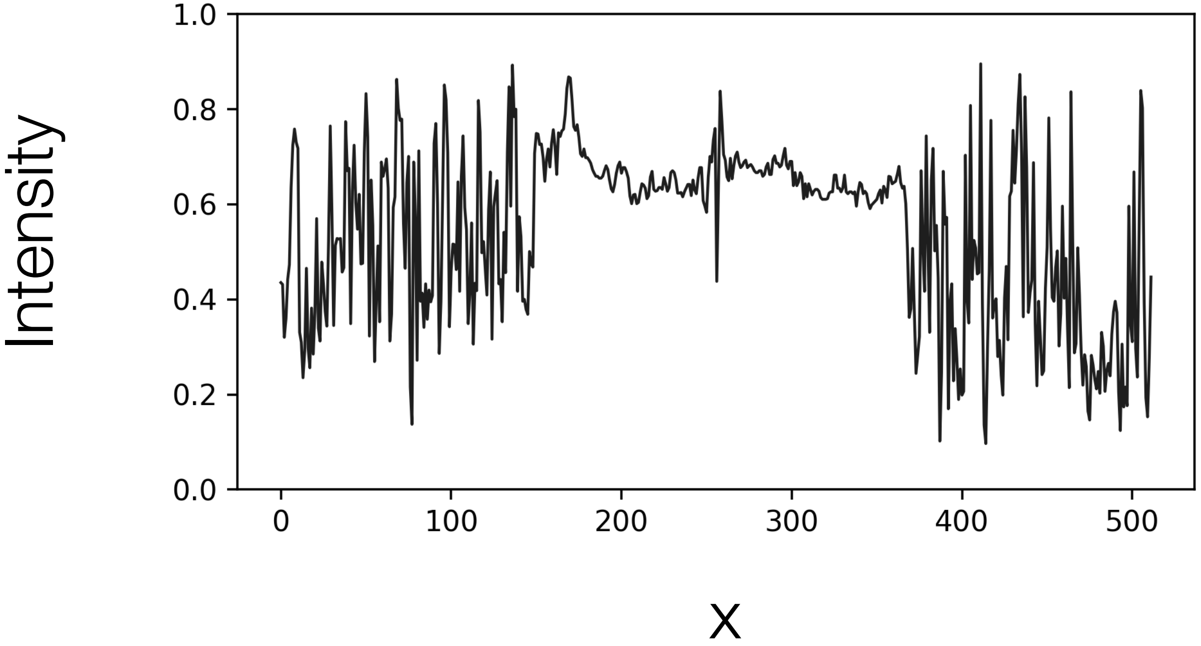
1D Fourier Transform and Images



1D Fourier Transform and Images

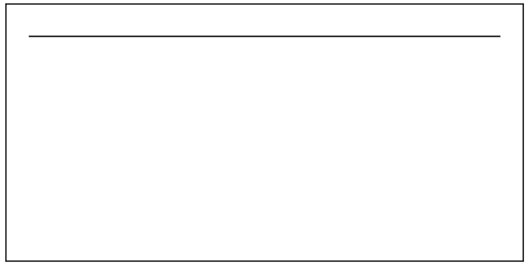
Image row



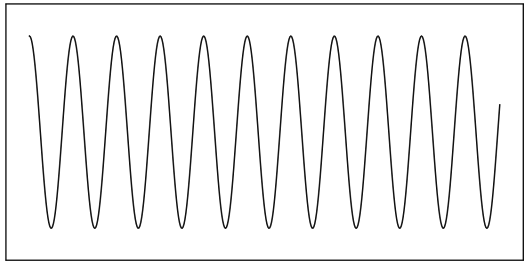


$=$

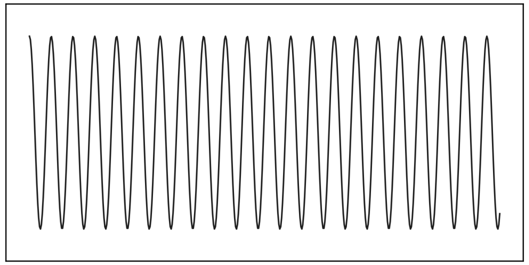
$F_1 \times$



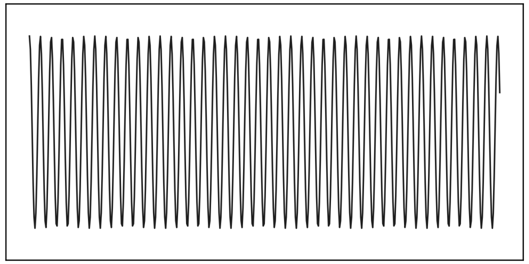
$+ \ F_2 \times$



$+ \ F_3 \times$

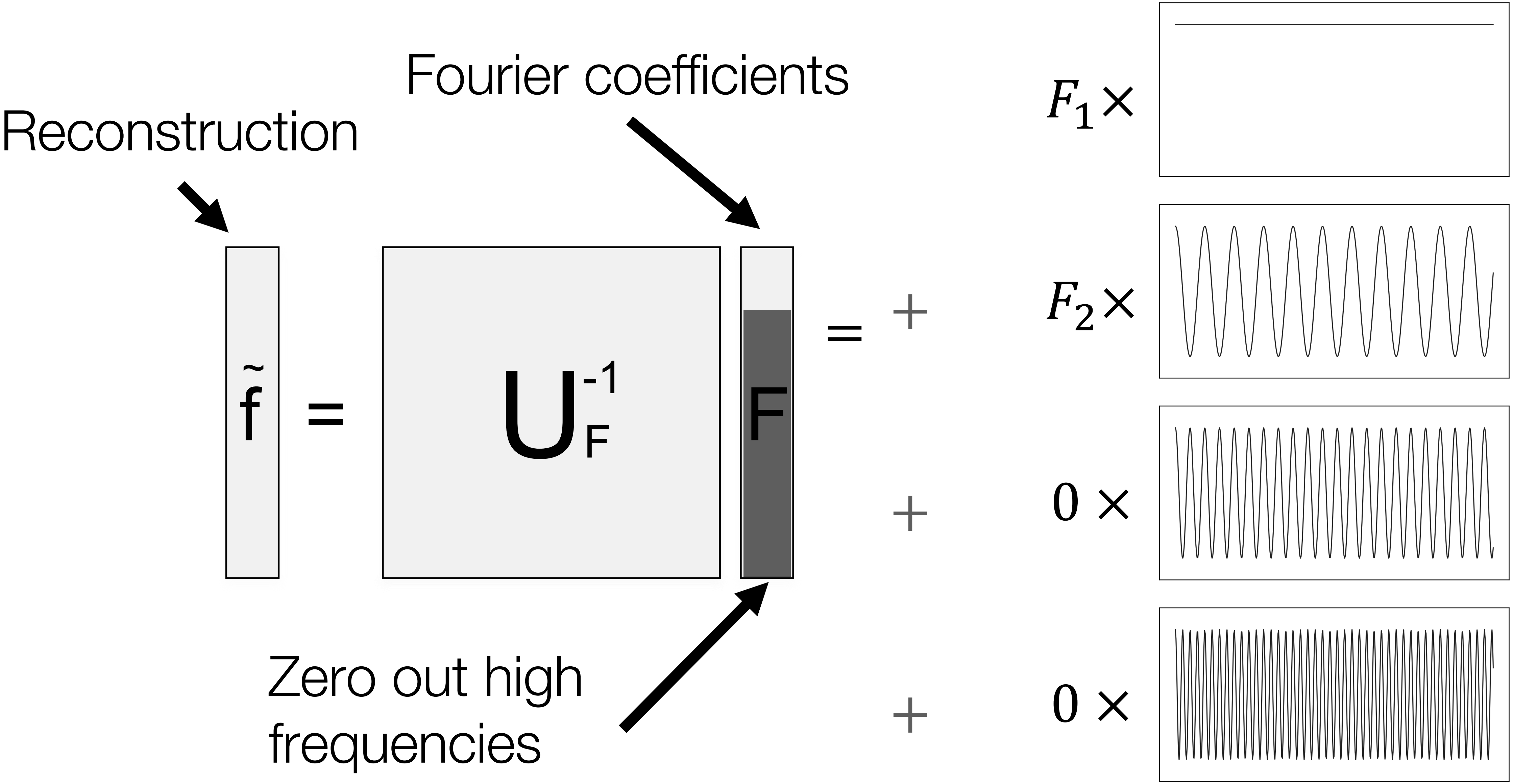


$+ \ F_4 \times$



Represent this function in a Fourier basis.

1D Fourier Transform and Images



Reconstructions of Different Freq.