Linear Independence

• A set of vectors are linearly dependent if you can write one as a linear combination of the others

• Suppose: \( \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \)

\[
x = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 2\mathbf{a} \quad y = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2} \mathbf{a} - \frac{1}{3} \mathbf{b}
\]

Is the set \{a, b, c\} linearly independent?

Is the set \{a, b, x\} linearly independent?
Basis

- Consider all vectors in $\mathbb{R}^3$ (3D Plane)
- A set of **linearly independent vectors** whose **span is the whole 3D plane** are called the basis for the 3D plane

E.g., the standard basis $\{i, j, k\}$ spans the whole 3D plane:

Any other vector in the plane (e.g., $a$) is a linear combination of $\{i, j, k\}$
Using Basis for expressing vectors

Example:

\[
\begin{bmatrix}
3 \\
2 \\
5
\end{bmatrix}
= 3 \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
+ 2 \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
+ 5 \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
2 \\
5
\end{bmatrix}
\]

\{i, j, k\} are the three basis vectors here

We could decompose it in terms of some other basis as well
Intuition behind Fourier transform: change of basis

\[ \vec{f} = \vec{e}_1 \cdot F(0) + \vec{e}_2 \cdot F(1) + \vec{e}_3 \cdot F(2) \]

where

\[ \vec{f} \triangleq f(n), \vec{e}_1 \triangleq \frac{1}{3} e^{2\pi in*0}, \vec{e}_2 \triangleq \frac{1}{3} e^{2\pi in*1}, \vec{e}_3 \triangleq \frac{1}{3} e^{2\pi in*2} \]

\[ \vec{f} = \begin{bmatrix} F(0) \\ F(1) \\ F(2) \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} \vec{f} \end{bmatrix} \]

Bottom line:

- Fourier coefficients are coordinates in the Fourier basis defined by \( \vec{e}_1, \vec{e}_2, \vec{e}_3 \)
- Calculating Fourier coefficients is just about finding the projection on the vector \( f(n) \) along the basis
Discrete Fourier Transform

We can extend this to any vector of length N:

\[ F[u] = \sum_{n=0}^{N-1} f[n]e^{(-2\pi i \frac{un}{N})} \]

where \( e^{(-2\pi i \frac{un}{N})} = \cos \left( 2\pi i \frac{un}{N} \right) - i \sin \left( 2\pi i \frac{un}{N} \right) \)

• Output F is a weighted sum of sines and cosines with the weights governed by input f
• We can think of the exponentials as basis functions, and the function F is expressed in terms of those basis
Continuous Fourier Transform

Going from continuous to discrete just means we take the integral from \(-\infty\) to \(\infty\):

\[
X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t} dt
\]
Time domain to frequency domain

• Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies
  
i.e., if we weighted sum across the different frequencies, we reconstruct the original signal

Source: A.Efros
Frequency Basis

We’re using a basis of sinusoids with different frequencies.
Complex Exponential Review

\[ \vec{f} \triangleq f(n), \quad \overrightarrow{e_1} \triangleq \frac{1}{3} e^{2\pi i n^0_3}, \quad \overrightarrow{e_2} \triangleq \frac{1}{3} e^{2\pi i n^1_3}, \quad \overrightarrow{e_3} \triangleq \frac{1}{3} e^{2\pi i n^2_3} \]
Visualizing Fourier Transform Matrix

\[ \exp \left( -2\pi i \frac{un}{N} \right) \quad \text{For } N=16 \]

When \( u = 0 \), \( \exp \left( -2\pi i \frac{un}{N} \right) = \exp(-2\pi i) \) for all \( n \)

\( \Rightarrow \) no change in frequency from \( n = 0 \) to \( n = N \)

Source: Torralba, Freeman, Isola
Visualizing Fourier Transform Matrix

\[ \exp \left( -2\pi i \frac{un}{N} \right) \]  

For \( N = 16 \)

When \( u = 1 \), \( \exp \left( -2\pi i \frac{un}{N} \right) = \exp \left( -2\pi i \frac{n}{N} \right) \) for all \( n \)

\( \implies \) no change in frequency from \( n = 0 \) to \( n = N \)

Source: Torralba, Freeman, Isola
Visualizing Fourier Transform Matrix

\[ \exp\left(-2\pi i \frac{un}{N}\right) \] For \( N=16 \)

Source: Torralba, Freeman, Isola
Examples

Let’s say $a = [1, 0, 0, 0]$, $N = 4$

$$F[u] = \sum_{n=0}^{N-1} f[n]e^{-2\pi i \frac{un}{N}} \quad (u = 0, 1, ... N - 1)$$
Examples

Let’s say \( a = [1, 0, 0, 0] \), \( N = 4 \)

\[
F[u] = \sum_{n=0}^{N-1} f[n] e^{-2\pi i \frac{un}{N}} \quad (u = 0, 1, \ldots, N - 1)
\]

\[
F[u] = f[0] e^{-2\pi i \frac{u \cdot 0}{4}} + f[1] e^{-2\pi i \frac{u \cdot 1}{4}} + f[2] e^{-2\pi i \frac{u \cdot 2}{4}} + f[3] e^{-2\pi i \frac{u \cdot 3}{4}} = f[0] = 1
\]

\[a = \text{np.array}([1, 0, 0, 0])\]
\[\text{np.fft.fft}(a)\]

\[\text{array([1.+0.j, 1.+0.j, 1.+0.j])}\]

All coefficients are 1!
1D Fourier Transform and Images
Represent this function in a Fourier basis.
1D Fourier Transform and Images

Reconstruction

\[
\tilde{f} = U_F^{-1} F
\]

Zero out high frequencies

\[
F_1 \times + F_2 \times + 0 \times + 0 \times
\]
Reconstructions of Different Freq.